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Synchronization of coupled oscillators Kuramoto model

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Abstract

La presente tesi è volta a spiegare e concettualizzare il modello e l'analisi svolta da Kuramoto, così come lo studio della stabilità del modello sotto varie circostanze. La tesi inizia contestualizzando il background storico. Il modello di Kuramoto viene poi introdotto e analizzato per distribuzioni di oscillatori discrete e continue. Alcune importanti caratteristiche sono dimostrate e spiegate, come la divisione degli oscillatori in due classi e l'esistenza di condizioni che permettano la sincronizzazione. Infine sono discusse la stabilità del modello, le condizioni per ottenere il caos nel caso finito e l'impossibilità di ottenere il caos nel modello continuo.

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Chapter 1

Introduction

Tutti coloro che abbiano mai partecipato ad un concerto, una conferenza o uno spettacolo teatrale avranno espresso il loro apprezzamento, battendo le mani insieme al resto del pubblico. Ascoltando attentamente, l'applauso, inizialmente casuale, converge velocemente ad una sincronizzazione ordinata di battiti di mani[1]. Questo esempio è uno dei molti sistemi in cui si osserva il fenomeno della sincronizzazione spontanea per un insieme di oscillatori. La sincronizzazione è un processo che accade quando due o più oscillatori interagiscono tra loro. Infatti, essi inizieranno ad oscillare con una frequenza approssimativamente identica, nonostante le frequenze naturali fossero drasticamente differenti.
Altri esempi si trovano in fisica, dove l'effetto dell'accoppiamento è ben compreso e si trova in fenomeni quali l'oscillazione di neutrini[2], gli oscillatori in giunzioni Josephson[3] e molti altri. Anche in biologia la sincronizzazione è osservata, per esempio nei network di cellule pacemaker[4]. Altri esempi ancora sono il suono delle cicale[5], alcune specie di lucciole che luccicano ad un ritmo comune[6], e molti altri. Anche quando l'accoppiamento di un sistema non è ben compreso o definito, la sincronizzazione avviene comunque e il modello descritto in questa tesi funziona adeguatamente. Una volta che il modello viene compreso, può essere applicato attraverso la tecnica del model checking[7]. Colui che contribuì in maggior misura allo studio della sincronizzazione fu Yoshiki Kuramoto, il cui modello è descritto in questa tesi.

Chapter 2

Preliminary

The first mention of synchronisation was done back in the 17^{th} century by Christian Huygens[8]. It was Wiener[9] who began the mathematical exploration in the field of collective synchronisation. He sought to approach the problem using Fourier Analysis but was unfortunately to no avail.
In 1967, Winfree[10] took another few at the problem. He examined the behaviour of many coupled oscillators. In his model, he included a number of simplifications. Firstly, the distribution of natural frequencies of the oscillators was assumed to be thin. (i.e. the oscillators would be nearly identical). He also assumed the coupling between the oscillators to be weak. Using these constraints, Winfree was able to approximate the oscillators to be coupled to some 'mean-field' oscillation of the collective system. He came up with his general model for the limit-cycle oscillators[1]

$$\dot{\theta}_i = \omega_i + \left(\sum_{j=1}^{N} X(\theta_j) \right) Z(\theta_i); \qquad i = 1, ..., N. \tag{2.1}$$

Equation (2.1) describes the frequency (or time derivative of the phase) $\dot{\theta}_i$ of the i^{th} oscillator. Which is given by its natural frequency ω_i plus some additional term that takes into account the coupling to each of the remaining oscillators. This additional term includes a function that gives the coupling strength $X(\theta_j)$ of the j^{th} oscillator to the collective system, as well as the coupling sensitivity $Z(\theta_i)$ of the i^{th} oscillator. Note that for the model to make sense, $X(\theta_j)$ and $Z(\theta_i)$ are required to be 2π-periodic functions on C^2[11].

[1] The limit-cycle oscillator is an attractor that describes the limiting behaviour of an oscillator.

Chapter 3

Kuramoto model

Kuramoto developed his analysis as a much more extensive continuation to the work of Winfree[10]. Intuitively, he realised that one could take the sensitivity term from Winfree's model into the sum and then express the model using a single coupling function Γ_{ij}. This coupling function then describes the coupling between the i^{th} and j^{th} oscillator, such that[12] the model becomes

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} \Gamma_{ij}(\theta_j - \theta_i), \qquad i = 1, ..., N. \tag{3.1}$$

Though, with the network topology being unspecified[9], the system of equations in the form of equation (3.1) is still far too difficult to investigate. However, intuitively, Kuramoto saw that the model could be understood by approximating the oscillators to be coupled to some mean field oscillation of the collective system. For his analysis, he then choose a purely sinusoidal coupling denoted by some coupling parameter K. Bearing these considerations in mind, Kuramoto wrote down his model where

$$\Gamma_{ij}(\theta_j - \theta_i) = \frac{K}{N} \sin(\theta_j - \theta_i). \tag{3.2}$$

The term N in equation (3.2) makes sure the system does not blow up as the number of oscillators becomes very large.
Additional motivation for the use of a sinusoidal coupling includes the idea that the coupling function would need to be odd if synchronisation is to occur. Effectively, what Kuramoto wrote down is simply a version of the first term in a Fourier series expansion of the Γ_{ij} function.
The natural frequencies of the oscillators ω_i would form some distribution that is described by some probability density function $g(\omega)$. Kuramoto assumed this

distribution to be unimodal[1] and symmetric around some mean frequency Ω. By analysing the system from a mean-field perspective, the probability function becomes symmetric around zero (that is, $g(-\omega) = g(\omega)$). To make use of this mean-field perspective, one would need to apply the translations

$$\theta_i \to \theta_i - \Omega t,$$
$$\omega_i \to \omega_i - \Omega.$$

3.1 The Order Parameter

To quantify the behaviour of the system in a more simplified way, the order parameter was introduced. The order parameter describes the collective rhythm of the model[13] and is given by

$$re^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}. \tag{3.3}$$

If one would imagine drawing the phase of each oscillator on a complex unit circle, then the order parameter can be thought of as some center of mass. The term ψ then gives the mean-field phase and r gives a quantification for coherence. In figure 3.1, three graphics are shown to illustrate the order parameter. The blue points around the circle indicate the phase of the N oscillators, and the orange cross indicates the order parameter itself.

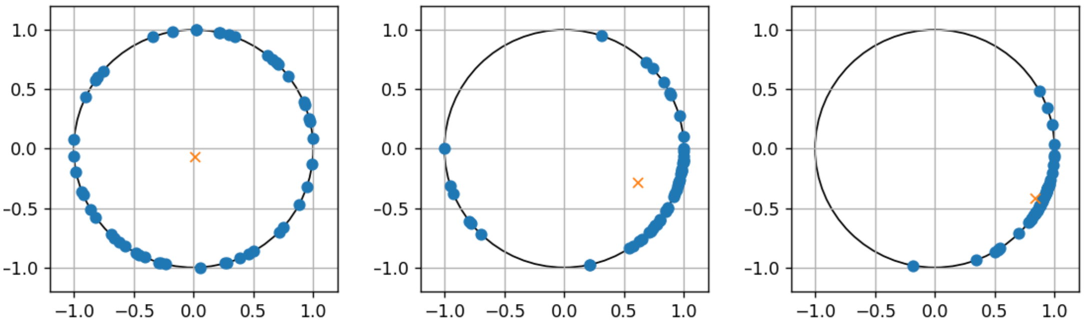

Figure 3.1: Visual representations of the order parameter.

It can be noted that the length of the order parameter r is always bounded between

[1]Unimodal functions monotonically decrease for $\omega > 0$. Formally: $\forall v \geq \omega, g(v) \leq g(\omega)$ on $[0, \infty)$.

0 and 1. When the order parameter is found close to the origin (i.e. $r \approx 0$), then the phases of the oscillators are mostly distributed on the full interval $[0,2\pi]$. A system where the order parameter approaches the origin is called *incoherent*. An incoherent system is (mostly) unsynchronised.
In contrast, if the order parameter is found near the unit circle (i.e. $r \approx 1$), then this means that the phases of the oscillators are mostly grouped together at one end. A system where the order parameter approaches the unit circle is called *coherent*. A coherent system is (mostly) synchronised.
For any other value of $0 < r < 1$, the system is said to be *partially synchronised*. This can be the case when some, but not all, oscillators lock to the collective rhythm. Further analysis on these types of solutions is done in section 4.1.

3.2 Further Simplification of the Kuramoto Model

Aside from being a handy tool to quantitatively examine a system of coupled oscillators, the order parameter can also be used to simplify the Kuramoto model given by the combination of equations (3.1) and (3.2). Taking the order parameter from equation (3.3), and multiplying left and right by some factor $e^{-i\theta_i}$ gives

$$re^{i(\psi-\theta_i)} = \frac{1}{N}\sum_{j=1}^{N} e^{i(\theta_j-\theta_i)}.$$

Equating the imaginary parts then gives

$$r\sin(\psi-\theta_i) = \frac{1}{N}\sum_{j=1}^{N} \sin(\theta_j-\theta_i).$$

Finally, substituting this result into equations (3.1) and (3.2) gives a much nicer version of the Kuramoto model:

$$\dot{\theta}_i = \omega_i + Kr\sin(\psi-\theta_i), \qquad i = 1, ..., N. \tag{3.4}$$

Using this form of the Kuramoto model, it is much simpler to recognise the dependence on the mean-field quantities r and ψ. The coupling term $Kr\sin(\psi-\theta_i)$ tends to bring the i^{th} phase closer to the mean-field phase ψ, whilst the effective coupling strength is proportional to the coherence term r.
One last simplification to the Kuramoto model can be obtained by considering a rotating frame. This would set $\psi = 0$ and results in a mean-field perspective:

$$r = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j} \tag{3.5}$$

$$\dot{\theta}_i = \omega_i - Kr\sin\theta_i, \qquad i = 1, ..., N. \tag{3.6}$$

3.3 Continuum Model

When considering a large number of oscillators, (i.e. $N \to \infty$), one could rephrase the distribution of frequencies $\dot{\theta}_i$ in terms of some probability density function $\rho(\theta, t, \omega)$. Then, $\rho(\theta, t, \omega)d\theta$ gives the fraction of oscillators whose phase lies between θ and $\theta + d\theta$ at time t[9]. Two conditions are imposed on the distribution. Namely that the distribution is normalised ($\int_0^{2\pi} \rho(\theta, t, \omega)d\theta = 1$), as well as it being periodic ($\rho(\theta, t, \omega) = \rho(\theta + 2\pi, t, \omega)$). The continuous order parameter is then given by

$$re^{i\psi} = \int_{-\pi}^{\pi} \int_{-\infty}^{+\infty} e^{i\theta'} \rho(\theta', t, \omega) g(\omega) d\omega d\theta' \tag{3.7}$$

By applying the same methods used for the discrete order parameter in section 3.2, it can be shown that

$$r \sin(\psi - \theta) = \int_{-\pi}^{\pi} \int_{-\infty}^{+\infty} \sin(\theta' - \theta) \rho(\theta', t, \omega) g(\omega) d\omega d\theta'.$$

Combining this with equation (3.4), one can rewrite the Kuramoto model in a continuous form as

$$\dot{\theta} = \omega + K \int_{-\pi}^{\pi} \int_{-\infty}^{+\infty} \sin(\theta' - \theta) \rho(\theta', t, \omega) g(\omega) d\omega d\theta'. \tag{3.8}$$

For the model to make sense, the time evolution of $\rho(\theta, t, \omega)$, which can be thought of as 'oscillator density' at certain phases has to be continuous. That is, the rate at which the oscillators leave a certain partition $d\theta$ of the unit circle in the order parameter has to be equal to the flux at the boundaries:

$$\frac{\partial \rho}{\partial t} + \frac{\partial [\rho \dot{\theta}]}{\partial \theta} = 0. \tag{3.9}$$

In physics, equation (3.9) resembles the conservation of linear momentum and mass. It originates from the same underlying principles. Since $\dot{\theta}$ is already known through equation (3.8), equation (3.9) can be rewritten as

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial \theta} \left[\rho \left\{ \omega + K \int_{-\pi}^{\pi} \int_{-\infty}^{+\infty} \sin(\theta' - \theta) \rho(\theta', t, \omega) g(\omega) d\omega d\theta' \right\} \right] \tag{3.10}$$

This partial integro-differential equation describes the Kuramoto model in the limit $N \to \infty$. It is non-linear and cannot be solved directly, but will prove to be very useful in the sections to follow. With this the governing equations for the Kuramoto model have all been addressed and it is time to start analysing them.

Chapter 4

Kuramoto Analysis

Numerical analysis on the connection between the coupling parameter K and the coherence term r shows a rather curious behaviour. Figure 4.1 shows the general trend of the relation between K and r[1]. As can be seen, there exists some critical value of the coupling parameter K_c, which marks the boundary between an incoherent and a (partially) coherent system. When $K < K_c$, the incoherent states seem to be the most stables once whereas when $K > K_c$, the coherent states become more and more stable. As K is increased beyond K_c more and more oscillators will tend to lock to the collective rhythm.
One can understand this from the Kuramoto model in equation (3.4). It can be noted that for synchronisation to occur, one would require a steady solution to some oscillator. That is, $\dot{\theta} = 0$. Imposing this condition to equation (3.6) and noting that r and sine are bound between 1 and -1 gives:

$$|\omega_i| \leq K, \qquad i = 1, ..., N. \tag{4.1}$$

So, to have any hope of achieving some form of synchronisation, the minimum requirement is that K must exceed some critical value for oscillator i.

4.1 Finite N Solution Terms

When the condition in equation (4.1) holds for some, but not all, oscillators in a system, the long-term solutions can be described by two terms:

[1]The exact relation heavily depends on the distribution of the natural frequencies $g(\omega)$. However, under the conditions imposed on $g(\omega)$ in section 3, the behaviour for (infinitely) many oscillators will, in general, look approximately the same.

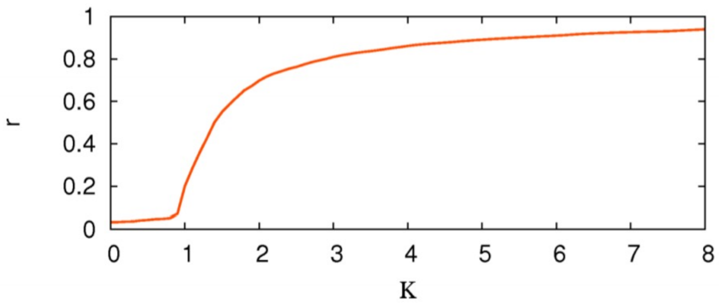

Figure 4.1: General plot of K against r.

- **Locked** oscillators ($|\omega_i| \leq Kr$): these are the oscillators that approach a stable limit-cycle as they do admit for the static solution $\dot{\theta}_i = 0$. In this case, it is possible to solve the Kuramoto model of equation (3.6) to get the phase distribution of locked oscillators around the mean phase:

$$\theta_i = \arcsin\left(\frac{\omega_i}{Kr}\right) \tag{4.2}$$

  Their natural frequencies ω_i are found near the center of the frequency distribution $g(\omega)$.

- **Drifting** oscillators ($|\omega_i| > Kr$): these are the oscillators that run around the unit circle non-uniformly. Although, at some occasions they seem to be affected by the synchronised terms, they will never be able to fully lock to the collective rhythm. This is because drifting terms do not allow for the solution $\dot{\theta} = 0$ and therefore possess no fixed points or attractors in θ. Their natural frequencies are found towards the tails of the frequency distribution $g(\omega)$.

4.2 Continuum Solution Terms

In the continuum limit, stationary states require the density distribution ρ to be independent of time, even when individual oscillators are still moving. Equation (3.9) then requires that ρ is inversely proportional to the angular velocity $\dot{\theta}$ as to keep their product constant. Combining this with equation (3.4) gives:

$$\rho(\theta, \omega) = \frac{C}{|\omega + Kr\sin(\psi - \theta)|} \tag{4.3}$$

where the constant C is found be

$$C = \frac{1}{2\pi}\sqrt{\omega^2 - (Kr)^2}$$

using the normalisation condition proposed in section 3.3.
Through self-consistency of the order parameter under the statements above, one can separately consider the locked and drifting oscillators. By denoting averages using angular brackets:

$$re^{i\psi} = \langle e^{i\theta} \rangle = \langle e^{i\theta} \rangle_{\text{lock}} + \langle e^{i\theta} \rangle_{\text{drift}}.$$

Considering the mean-field perspective ($\psi \to 0$):

- **Locked term:** using equation (3.7), and by noting that ρ does no longer explicitly depends on time:

$$\langle e^{i\theta} \rangle_{\text{lock}} = \int_{-\pi}^{\pi} \int_{-Kr}^{Kr} e^{i\theta'} \rho(\theta', \omega) g(\omega) d\omega d\theta'.$$

  By realising that the imaginary (sine) part of the exponent is anti-symmetric in the θ integral, and by including equation (4.2), the above can be recast into the form:

$$\langle e^{i\theta} \rangle_{\text{lock}} = \int_{-Kr}^{Kr} \cos\left(\arcsin\left(\frac{\omega}{Kr}\right)\right) g(\omega) d\omega.$$

- **Drifting term:** the contribution of the drifting oscillators can be given through equation (3.7):

$$\langle e^{i\theta} \rangle_{\text{drift}} = \int_{-\pi}^{\pi} \int_{|\omega|>Kr} e^{i\theta'} \rho(\theta', \omega) d\omega d\theta'.$$

  Through symmetry considerations in both $g(\omega)$ as well as $\rho(\theta, \omega)$, this integral vanishes.

In conclusion, the long term behaviour of the order parameter for infinitely many oscillators depends only on the locked term. The continuous order parameter can then be rewritten as:

$$\begin{aligned} r &= \int_{-Kr}^{Kr} \cos\left(\arcsin\left(\frac{\omega}{Kr}\right)\right) g(\omega) d\omega, \\ &= Kr \int_{-\pi/2}^{\pi/2} \cos^2(\theta) g(Kr \sin\theta) d\theta. \end{aligned} \tag{4.4}$$

Equation (4.4) represents the self-consistency condition of the Kuramoto model. There are two possible solutions: the trivial solution $r = 0$, which corresponds to the incoherent state $\rho(\theta, \omega) = \frac{1}{2\pi}$ for all θ, ω; and a solution for $r \neq 0$, in which case equation (4.4) reads:

$$1 = K \int_{-\pi/2}^{\pi/2} \cos^2(\theta) g(Kr \sin\theta) d\theta. \tag{4.5}$$

By letting $r \to 0^+$, one can find the inferior limit of K, which is the critical coupling value:

$$K_c = \frac{2}{\pi g(0)}.$$

Expanding equation (4.5) in powers of r around $r = 0$ gives:

$$\begin{aligned} 1 &= K \int_{-\pi/2}^{\pi/2} \cos^2(\theta) \left\{ g(0) + K_c \sin(\theta) g'(0) r + \tfrac{1}{2} K_c^2 \sin^2(\theta) g''(0) r^2 + ... \right\} d\theta \\ &\approx K g(0) \int_{-\pi/2}^{\pi/2} \cos^2\theta d\theta + \frac{r^2}{2} K K_c^2 g''(0) \int_{-\pi/2}^{\pi/2} \cos^2\theta \sin^2\theta d\theta, \\ &= K g(0) \frac{\pi}{2} + \frac{r^2}{2} K K_c^2 g''(0) \frac{\pi}{8}. \end{aligned}$$

From this, it can be concluded that the amplitude of the bifurcation branch is given by:

$$r \approx \sqrt{\frac{16}{\pi K_c^3} \frac{-\mu}{g''(0)}}, \tag{4.6}$$

where

$$\mu = \frac{K - K_c}{K_c}.$$

Chapter 5

Stability and chaos

This section will discuss several subtopics that come with stability of the Kuramoto oscillators. Firstly, stability in the continuum limit will be covered, where chaos *cannot* exist [14],[15]. That includes stability in the $r = 0$ branch as well as stability in the bifurcation in $K \geq K_c$, as mentioned in section **3**. Following that the finite N Kuramoto model and its chaos conditions and will be discussed.

5.1 Stability in the $N \longrightarrow \infty$ Limit

5.1.1 General Case

Both the $r = 0$ branch and the bifurcating branch of the continuum limit requires the usage of equations (3.9) and (3.10), which govern the dynamics of the oscillator density $\rho(\omega, \theta, t)$ and therefore also the whole system. In order to look for stability, one must now look at the fixed points and discuss them individually. To obtain the fixed points, the steady state ($\frac{\partial \rho}{\partial t} = -\frac{\partial[\rho\dot{\theta}]}{\partial \theta} = 0$) solutions need to be found. Evidently, this requires $\dot{\theta}\rho(\omega, \theta) = C(\omega)$ and $C \in \mathbb{R}$. However, now there are two cases that need to be considered, depending on whether the order parameter takes a nonzero value or not.

The case $r \neq 0$: The bifurcating branch

In the case that $r \neq 0$, there are two different and distinct solutions:

- $C(\omega) = 0 \implies \dot{\theta}\rho(\omega, \theta) = 0$ and since $\rho \neq 0$ due to the normalisation condition as in section **2.2**, it must be true that $\dot{\theta} = \omega - Kr\sin(\theta) = 0$. Inspired by the Kuramoto analysis it is checked for a solution the case with all locked oscillators. As seen in previous sections, this equation gives the attractor in θ for the locked oscillators with a frequency ω. This attractor is namely $\theta_0 = \arcsin(\omega/(Kr))$. This hints at the fact that a delta function solution of the form $\rho(\omega, \theta) = \rho(\theta) = \delta(\theta - \theta_0)$ needs to be checked. It turns out that this is indeed a solution and it really corresponds to the locked fraction of the oscillators.

- $C(\omega) \neq 0 \implies \rho(\omega, \theta) = \frac{C}{\dot{\theta}} = \frac{C}{\omega + Kr\sin(\psi-\theta)} = \frac{1}{2\pi}\frac{\sqrt{\omega^2-K^2r^2}}{|\omega+Kr\sin(\psi-\theta)|}$ is equivalent to equation (4.3) when setting $\psi = 0$ without loss of generality, which represents the distribution of the drifting population, which has no attractors in θ. The function $\rho(\omega, \theta)$ is almost uniform for large $|\omega|$, but when the frequency decreases, peaks start to appear at $\theta = \frac{\pi}{2}$ and $\theta = -\frac{\pi}{2}$ for positive and negative ω respectively, meaning that oscillators spend significantly more time around these phases.

Understanding why that is the case can give a lot of insight into the behaviour of the Kuramoto model. The peaks are precisely at these values because those values are the maximum phase at which an attractor can exist. In other words, when $|\omega|$ gets lowered beyond some threshold, the oscillator becomes locked and it is first locked at $\theta_0 = \pm\frac{\pi}{2}$. Further decreasing $|\omega|$ moves the attractor towards the mean phase ($\psi = 0$).

Therefore, it is now clear that Kuramoto's intuition for two separate populations in the system is correct and why it makes sense. The drifting and the locked oscillators each go to their asymptotic values to form a total order parameter r. Near the bifurcation r behaves as is written in equation (4.6) **if** the distribution $g(\omega)$ is supercritical (i.e. $g''(\omega) < 0$). Even though there is no analytic solution for $r(K)$ for the full branch, it is in general true that chaos does not exist for the case of a pure Kuramoto model in the thermodynamical limit $N \to \infty$ [14]. This result is unanimously confirmed by numerical simulations [9]. It is, however, still desired to have a more concrete proof that the two solutions above are stable.

The case $r = 0$: The incoherent state

If the order parameter is allowed to approach a zero value, then it can be easily seen that the attractors for the locked oscillators start disappearing even for the smallest of frequencies. Meanwhile, the drifting oscillators start being distributed more and more uniformly around the unit circle. The state when $r = 0$, leading to solution $\rho = \frac{1}{2\pi}$, is called the incoherent state. It is definitely a fixed point of the system, but its stability is still undetermined. Strogatz tackled this problem in the following way [16]. The incoherent state is perturbed so that

$$\rho(\omega, \theta, t) = \frac{1}{2\pi} + \epsilon\eta(\omega, \theta, t) \quad \text{with} \quad \epsilon \ll 1,$$
$$\eta(\omega, \theta, t) = c(\omega, t)e^{i\theta} + \tilde{c}(\omega, t)e^{-i\theta} + \eta^{+}.$$

The question to ask now is how does $\rho(\omega, \theta, t)$ evolve in time and does it collapse back to the incoherent state or does it develop into chaotic dynamics? More specifically, does $c(\omega, t)$ diverge or converge? To answer that question, one can let

$c(\omega, t) = b(\omega)e^{\lambda t}$ and substitute that and the whole expression from the equation above into the governing equation (3.10) in order to analyse for λ. Since the incoherent state is a solution, the governing equation reduces to a dynamical system in c and it looks like

$$\frac{dc}{dt} = \hat{\mathcal{A}}c = \lambda c, \quad \text{where} \quad \hat{\mathcal{A}}c = -i\omega c + \tfrac{K}{2}\int_{-\infty}^{\infty} c(t, \omega')g(\omega')d\omega'. \tag{5.1}$$

Evidently, the operator $\hat{\mathcal{A}}$ is linear. After a few steps, it is found is that a solution for λ is unique and real and with this condition, in must be true that

$$1 = \tfrac{K}{2}\int_{-\infty}^{\infty} \frac{\lambda\, g(\omega)\, d\omega}{\lambda^2 + \omega^2}. \tag{5.2}$$

Since all expressions except for λ in equation 5.2 are positive, then λ must be non-negative, which means that the perturbation either forces the incoherent state into neutral stability, or chaos. It turns out that the condition that separates these two outcomes is the coupling parameter K so that the incoherent state is *linearly*[1] *neutrally stable*[2] for $K < K_c$ and as expected - unstable for $K \geq K_c$. This result is reconfirmed in more recent papers as well for the nonlinear [17] and general case [18].

As for the bifurcating branch, so far it can only be shown that the order parameter is stable in the neighbourhood of the critical point K_c. As for the whole branch, stability in the general case is not yet proven, even though simulations hint at it.

5.1.2 Lorentzian-like Distributions

So far we have discussed only the general case of an even and unimodal distribution of natural frequencies $g(\omega)$. But what if we pick a more specific distribution like the Lorentzian. Let $g(\omega) = \frac{(\gamma/\pi)}{\omega^2+\gamma^2}$, centred at $\Omega = 0$ as we are still in the rotating frame. Therefore, $K_c = \frac{2}{\pi g(0)} = 2\gamma$ Kuramoto himself [19] gave this as an example

[1] *Linear because the extra terms η^+ of $\eta(\omega, \theta, t)$ are ignored and thus $\hat{\mathcal{A}}$ is linear.*

[2] Point x_0 is neutrally stable in the dynamical system $(\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}; (t, x) \mapsto \phi_t(x)))$ if it is Lyapunov stable (i.e. for any open neighbourhood $\mathcal{U}$ of x_0, $\exists$ a smaller neighbourhood $\mathcal{V}$ of x_0, s.t. $\phi_t(\mathcal{V}) \subset \mathcal{U} \quad \forall t \geq 0$) and is not attracting.

and performed the calculation for r from equation (4.5):

$$\begin{aligned} 1 =& \frac{K}{\pi}\int_{\pi/2}^{\pi/2} \frac{\gamma\cos^2\theta d\theta}{\gamma^2 + K^2 r^2 \sin^2\theta}, \\ =& \frac{-\gamma + \sqrt{K^2r^2+\gamma^2}}{Kr^2}, \\ \Longrightarrow \qquad & r = 0 \quad \text{or} \quad r = \sqrt{1-\frac{2\gamma}{K}} = \sqrt{1-\frac{K_c}{K}}. \end{aligned}$$

This is not the only approach that can be used to reach this result. In fact, a different approach will be discussed now without too much detail, which allows for a more general conclusion and what is found above for the Lorentzian will serve as a reality check. The approach is done by Ott and Antonsen and its main result is that for a general distribution $g(\omega) = \frac{A}{\omega^{2m}+B^2},\ A, B \in \mathbb{R}_+$ the Kuramoto model reduces to m coupled differential equations [20].In short, the way in which this result is obtained is by expanding ρ in Fourier series of θ such that

$$\begin{aligned} \rho(\omega,\theta,t) &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} f_n(\omega,t)e^{in\theta}, \\ &= \frac{1}{2\pi}\left(1+\sum_{n=1}^{\infty} f_n(\omega,t)e^{in\theta} + \tilde{f}_n(\omega,t)e^{-in\theta}\right) \end{aligned} \tag{5.3}$$

and looking for the specific family of solutions $f_n(\omega,t) = (\alpha(\omega,t))^n$ with $|\alpha| \leq 1$.[3] Since equations (3.7) and (3.10) must be satisfied, this leads to

$$\frac{\partial\alpha(\omega,t)}{\partial t} = \frac{K}{2}\left(\underbrace{\left[\int_{-\infty}^{\infty}\alpha(\omega,t)g(\omega)d\omega\right]}_{r^*} - r\alpha^2\right) - i\omega\alpha(\omega,t).$$

Meanwhile, the Fourier series from equation 5.3 reduce to geometric series and allow for a concrete expression for $\rho(\omega,\theta,t)$. Now, to obtain the equation for the order parameter, we must integrate $r = \int_{-\pi}^{\pi} d\theta \int_{-\infty}^{\infty} d\omega\rho g$ (equation (3.7)), but before doing so, we need to find the expression for r^*. That integral is computed by making use of a semi-circle contour in the complex plane, which would enclose exactly half of the poles of $g(\omega)$. Using Cauchy's residue theorem and a useful lemma[4], there can be found m solutions for $\alpha(\omega,t)$. This condition transforms into m coupled differential equations for r.

[3] One motivation to consider these solutions is the fact that both the incoherent state ($\alpha = 0$) and the partially synchronised states conform to that expression. Therefore, this family of distributions can be interpreted as a continuous connection between the two extrema.

[4] When the integrand is of the form $I = \frac{P_1}{P_2}$ (polynomials), then the integral vanishes along the open semi-circle contour in the limit that the radius goes to infinity if $\deg(P_1)+2 \leq \deg(P_2)$

In the simplest case where $g(\omega) = \frac{(1/\pi)}{\omega^2+1}$, this method results in the equation $\dot{r} = (\frac{K}{2} - 1)r - \frac{K}{2}r^3$ and its stationary state corresponds to the result obtained above. For $K > K_c$ his equation has two fixed points $r_1 = 0$ and $r_2 = \sqrt{1 - \frac{2\gamma}{K}}$, only the second one of which is an attractor. Therefore the omega and alpha limit sets are $\omega^\dagger = (r_2)$ and $\alpha^\dagger = (r_1)$. For $K \leq K_c$ the fixed point is only one ($r_1 = 0$) and it is attracting, thus having only an omega limit set of $\omega^\dagger = (r_1)$ and an empty alpha set $\alpha^\dagger = \varnothing$.

5.2 Stability for $N < \infty$

5.2.1 Identical Oscillators

The case of identical oscillators is the simplest possible first look at the problem of stability in a finite-dimensional Kuramoto model. In a rotating frame with the frequency of the oscillators, the problem is simply formulated like

$$\dot{\theta}_i = \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i) = -Kr\sin\theta \qquad (5.4)$$

where the last step is only valid if the mean phase ψ is set to zero again. This system possesses two fixed points if $r \neq 0$ - at $\theta = 0$ and $\theta = \pm\pi$ (same point). Only the first of these is attracting and is asymptotically stable. The existence of such a global attractor forbids chaos in the system. The equivalent of the incoherent state (e.g. $\theta_\nu = \frac{2\pi}{N}(\nu - 1)$) here can also be seen to be non-chaotic and an equilibrium point.

Alternatively, one can reduce the system of N equations 5.4 into a $(N-1)$-dimensional dynamical system by the method of averaging[5]. It can be shown that this system has at least $N - 2$ constants of motion [21]. Therefore, the dynamical system from equation 5.4 reduces to a *one-dimensional* dynamical system. As is known, planar chaos cannot exist and even less so one-dimensional chaos. Therefore, chaos cannot exist in the case of identical oscillators.

5.2.2 Non-identical Oscillators

In the cases $N = 2,3,4$

[5] Consider $N - 1$ equations for phase differences and one for the mean phase, which does not couple back to the phase differences.

In the simplest case when $N = 2$, the system is not chaotic. As explained above, the method of averaging can be used to reduce the system to one equation, such that

$$\dot{\phi} = \Delta - K \sin \phi \quad \text{where} \quad \phi = \theta_2 - \theta_1 \quad \text{and} \quad \Delta = \omega_2 - \omega_1. \tag{5.5}$$

Once again, chaos does not exist in one dimension and this system is very well behaved. What's more, the system still synchronises to a locked state if $\dot{\phi} < 0$. The condition for this is that equation 5.5 has two fixed points and therefore the condition is $K > K_c = \Delta$. In the case $K < K_c$, the two oscillators oscillate normally, but with altered frequencies $\bar{\omega}$, given by [22]:

$$\bar{\omega}_{1,2} = \frac{\omega_1 + \omega_2}{2} \pm \pi \left(\int_0^{2\pi} \frac{d\phi}{\Delta - K \sin \phi} \right)^{-1}. \tag{5.6}$$

The case of $N = 3$ is the first one that makes the dynamics a bit less predictable. Since the system can still be reduced to two dimensions, it should not exhibit chaos either. There still exists a coupling constant K_c past which oscillations become locked at a common frequency [22]. It is possible that under uneven distribution of frequencies, an additional coupling constant K_{cl} appears past which two of the oscillators synchronise. In general, the system has six fixed points - one stable, two unstable and three saddles which start disappearing when K is reduced past the bifurcation point(s) [22],[23].

If one more oscillator is added to the picture so that $N = 4$, this is the first time that true chaos appears. As can be seen, the argument for absence of chaos due to low dimension of the system no longer holds. Furthermore, it can be proven that a chaotic attractor emerges in this case [22]. There is again a point K_c above which the system is synchronised. However, this time reducing K through K_c results in chaos and not quasi-periodic dynamics.

In the general case for larger N

It becomes fairly hard to analyse equations as N increases past 10. Nevertheless, in all cases of finite N, the point K_c always exists (increasing at first with N, but decreasing and converging to a positive value later) and past it there is always an asymptotically stable value for r with a certain upper bound, also increasing with N [24]. Chaotic behaviour increases with N, but that is only up to a certain point. As can be intuitively explained, large N slowly approaches the system with infinite N, which as already shown is non-chaotic [14]. For very large, but finite N, it can be shown that there exists a threshold value for the width of the frequency distribution, below which stable phase-locked solutions exist [25].

Chapter 6

Conclusion

The Kuramoto model, firstly introduced in 1975 by Yoshiki Kuramoto during a symposium, has come to represent the baseline of what is nowadays an entire sub-field of mathematics, dealing with synchronisation of oscillators. As seen previously, this model finds applications in many fields outside mathematics. The simplest possible case is the unperturbed infinite-oscillator model, which allows for some simplified calculations. However, most of the progress that could be done there was done by the mid-90s. That is - the existence of two groups of partially synchronised oscillators past a certain threshold coupling strength and the stability of the incoherent state and the partially synchronised around the bifurcation. Additionally, computer simulations have also been quite handy through the years of researching the Kuramoto model and predicting its behaviours under various conditions. But still, since the 90s there haven't been huge developments in the 'classic' Kuramoto model and the problem concerning global stability was left hanging. Hence, other researchers in the field turned to discussing finite-oscillator models and their size dependence. Furthermore, many ventured into driven oscillators under the Kuramoto model, different distinct groups of oscillators composing the system, etc.

Appendix A

Numerical integration with C++

```
// Kuramoto model (Synchronization of N oscillators)
// headers from Numerical Recipes (3rd edition)
#include <iostream>
#include <iomanip>
#include <cmath>
#include <random>
#include <complex>
#include "nr3.h"
#include "stepper.h"
#include "stepperdopr5.h"    // fifth-order Dormand-Prince method
#include "odeint.h"


using namespace std;

const int n = 4;          // number of oscillators
double omega_array[n];        // omega's array
int time_int = 1200;          // integration time
double K = 0.2;           // coupling parameter
int counts = 0;

// set of n differential equations
struct derivates{
    derivates(){};
    void operator()(const double& t, const VecDoub &yvector, VecDoub
   &dydx){
        int i = 0;
        int k = 0;
```

```
        counts++;

        for(i = 0; i < n; i++){
            dydx[i] = omega_array[i];
            for(k = 0; k < n; k++){
                dydx[i] += K/n * sin(yvector[k] - yvector[i]);
            }
        }       // 2 for loops in order to realize Kuramoto model
    }
};
Doub zero_2_PI(Doub x){      // phases between 0 and 2pi
    while(x <= 0){
        x += 2.0*M_PI;
    }
    while(x > 2*M_PI){
        x -= 2.0*M_PI;
    }return x;
}

int main(){
    const int N = n;
    Doub x1 = 0.0, x2 = 0.0, dx = 0.003;     // integration boundaries, stepsize
    Doub atol = 1e-12, rtol = 1e-12, stepmin = 1e-18;    // abs e rel tolerances, min stepsize
    Doub x1max = time_int * dx;      // stepsize * integration time
    VecDoub ystart(N);                    // initial states array
    int i = 0;
    ofstream omega("omega.csv");     // write omegas on omega.csv
    ofstream phase("phases.csv");    // write phases on phases.csv
    std::default_random_engine gen;     // random normal distribution of frequencies
    std::normal_distribution<double> norm(2.5,1.0);
    for(i=0; i<n;i++){
        omega_array[i]=norm(gen);
        omega << omega_array[i] <<" "<<endl;
    }
    std::default_random_engine gen2;             // uniform distribution of phases between 0 and 2pi
    std::uniform_real_distribution<double> unif(0,2*M_PI);
    for(i=0; i<n;i++){
        ystart[i]=unif(gen2);
        phase<< ystart[i] <<" "<<endl;
    }
    omega . close();
    phase . close();
    counts = 0;
    ofstream trajectory("trajectory.csv");
    ofstream order_1("order.csv");
```

```
    ofstream order_2("Rorder.csv");
    for( x1 = 0; x1 < x1max; x1 += dx){
        x2 = x1 + dx;    // int. boundaries are x1 and x1+dx=x2
        cout << "time: " << x1 << " " << x2 << endl;
        derivates d;
        Output out(10); // Output is an object that takes intermediate values
        Odeint <StepperDopr5<derivates> > ode(ystart, x1, x2, atol, rtol ,dx ,stepmin , out, d);
        ode.integrate();     // integration of differential eqs.
        for(i = 0; i < n; i++){
            ystart[i] = zero_2_PI(ystart[i]);    // phases between 0 and 2pi
        }
        trajectory << x1;    // lower bound printed on trajectory.csv
        complex<double> im = -1;
        im = sqrt(im);                  // imaginary unit
        for(i = 0; i < n; i++){
            trajectory << "," << ystart[i] ;     // new phases printed on trajectory
        }
        trajectory << endl;
        complex<double> order = exp(im * ystart[0]);     // order parameter
        for(i = 1; i < n; i++){
            order += exp(im * ystart[i]);
        }
        double realorder = real(order) / n;      // real part
        double imagorder = imag(order) / n;      // imaginary part
        double order1 = sqrt(realorder * realorder + imagorder * imagorder);     // module
        order_1 << x1 << ","<< realorder << ","<< imagorder <<endl;
        order_2 << x1 <<","<< order1 << endl;
    }
    trajectory.close();
    order_1.close();
    order_2.close();
    return 0;
}
}
```

Appendix B

Animation with Python

```
import numpy as np
import matplotlib.pyplot as plt
import math
from matplotlib import animation
plt.style.use('seaborn-pastel')

#================
#=====DATA=====
#================
k=0.2                 #coupling parameter
n=4                   #number of oscillators
time_int=1200
dt=0.003


data = np.genfromtxt('trajectory.csv', delimiter = ',')     #data from trajectory.csv
oscill=np.array([np.array ([np.array ([0. ,1. ,0.]) for osc in range(n)]) for time in range(time_int)])
for t in range(time_int):
    for osc in range(n):
        oscill[t][osc][0]=data[t][osc+1]  #theta
        oscill[t][osc][1]=math.cos(data[t][osc+1]) #cos theta
        oscill[t][osc][2]=math.sin(data[t][osc+1]) #sin theta
#my array has [theta,cos(theta),sin(theta)] for each oscillator and for each instant (1200 rows, n columns)


data_order = np.genfromtxt('order.csv', delimiter= ',')        #data from order.csv e Rorder.csv
data_r_order = np.genfromtxt('Rorder.csv', delimiter= ',')
```

```python
order=np.array([np.array([0., 1., 0.]) for time in range(time_int)])
for x in range(time_int):
    order[x][0] = data_r_order[x][1]     #modulo
    order[x][1] = data_order[x][1]       #parte reale
    order[x][2] = data_order[x][2]       #parte immaginaria
#my array has [r module,Re(r),Im(r)] for each instant (r order parameter)


#==========
#==FIGURE==
#==========
fig = plt.figure()
ax = fig.add_subplot(autoscale_on=True, xlim=(-1.5, 1.5), ylim=(-1.5, 1.5))
fig.suptitle('K='+str(k)+', N='+str(n))
ax.set_aspect('equal')
ax.grid()
ax.set_xlabel('Re')
ax.set_ylabel('Im')
cir=plt.Circle((0 ,0) ,1,fill=False)  #complex unit circle
ax.add_artist(cir)
osc, =ax.plot([],[],'o')
time_template = 'time = %.1fs'
time_text = ax.text(0.05, 0.9, '', transform=ax.transAxes)  #module of r instead of time


#==========
#==ARROW===
#==========
Q=ax.quiver(0,0,0,0,units='xy',scale=1)  #arrow with coordinates Re(r) and Im(r)

#==========
#==INIT====
#==========
# initialization function
def init():
    osc.set_data([], [])
    time_text.set_text('')
    return osc, time_text


#=============
# = Animate =
#=============
def animate(i):
```

```
    osc.set_data([oscill[i][osc][1] for osc in range(n)],[oscill[i][osc][2] for osc in range(n)])
    Q.set_UVC([order[i][1]],[order[i][2]])
    time_text.set_text('r = ' + str(order[i][0]))
    return osc, Q, time_text

anim=animation.FuncAnimation(fig, animate, frames=righe, interval=8, blit=False)

plt.show()
anim.save('Kuramoto(k='+str(k)+',n='+str(n)+').mp4', fps=30)
```

Bibliography

[1] Zoltán Néda, Erzsébet Regan, Yves Bréchet, Tamás Vicsek, and Albert-Laszlo Barabasi. «The sound of many hands clapping - Tumultuous applause can transform itself into waves of synchronized clapping». In: *Nature* 403 (Feb. 2000), pp. 849–850. DOI: 10.1038/35002660 (cit. on p. 1).

[2] James T. Pantaleone. «Stability of incoherence in an isotropic gas of oscillating neutrinos». In: *Phys. Rev. D* 58 (1998), p. 073002. DOI: 10.1103/PhysRevD.58.073002 (cit. on p. 1).

[3] Kurt Wiesenfeld, Pere Colet, and Steven H. Strogatz. «Frequency locking in Josephson arrays: Connection with the Kuramoto model». In: *Phys. Rev. E* 57 (2 Feb. 1998), pp. 1563–1569. DOI: 10.1103/PhysRevE.57.1563. URL: https://link.aps.org/doi/10.1103/PhysRevE.57.1563 (cit. on p. 1).

[4] Daniel C. Michaels, E P Matyas, and José Jalife. «Mechanisms of sinoatrial pacemaker synchronization: a new hypothesis.» In: *Circulation research* 61 5 (1987), pp. 704–14 (cit. on p. 1).

[5] Kathy S Williams and Chris Simon. «The ecology, behavior, and evolution of periodical cicadas». In: *Annual review of entomology* 40.1 (1995), pp. 269–295 (cit. on p. 1).

[6] John Buck. «Synchronous Rhythmic Flashing of Fireflies. II.» In: *The Quarterly Review of Biology* 63.3 (1988), pp. 265–289. ISSN: 00335770, 15397718. URL: http://www.jstor.org/stable/2830425 (cit. on p. 1).

[7] Ezio Bartocci, Flavio Corradini, Emanuela Merelli, and Luca Tesei. «Detecting synchronisation of biological oscillators by model checking». In: *Theoretical Computer Science* 411.20 (2010), pp. 1999–2018 (cit. on p. 1).

[8] Christiaan Huygens. «Horologium Oscillatorium (The pendulum clock)». In: *Trans RJ Blackwell, The Iowa State University Press, Ames* (1986) (cit. on p. 2).

[9] Steven H. Strogatz. «From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators». In: *Physica D: Nonlinear Phenomena* 143.1 (2000), pp. 1–20. ISSN: 0167-2789. DOI: https://doi.org/10.1016/S0167-2789(00)00094-4 (cit. on pp. 2, 3, 6, 12).

[10] Arthur T. Winfree. «Biological rhythms and the behavior of populations of coupled oscillators». In: *Journal of Theoretical Biology* 16.1 (1967), pp. 15–42. ISSN: 0022-5193. DOI: https://doi.org/10.1016/0022-5193(67)90051-3 (cit. on pp. 2, 3).

[11] W Oukil, A Kessi, and Ph Thieullen. «Synchronization hypothesis in the Winfree model: Synchronization in Winfree model with N oscillators.» In: (2016), p. 3 (cit. on p. 2).

[12] Yoshiki Kuramoto. «International symposium on mathematical problems in theoretical physics». In: *Lecture notes in Physics* 30 (1975), p. 420 (cit. on p. 3).

[13] Hyunsuk Hong and Steven Strogatz. «Kuramoto Model of Coupled Oscillators with Positive and Negative Coupling Parameters: An Example of Conformist and Contrarian Oscillators». In: *Physical review letters* 106 (Feb. 2011), p. 054102. DOI: 10.1103/PhysRevLett.106.054102 (cit. on p. 4).

[14] Oleksandr V Popovych, Yuri L Maistrenko, and Peter A Tass. «Phase chaos in coupled oscillators». In: *Physical Review E* 71.6 (2005), p. 065201 (cit. on pp. 11, 12, 16).

[15] Christian Bick, Mark J Panaggio, and Erik A Martens. «Chaos in Kuramoto oscillator networks». In: *Chaos: An Interdisciplinary Journal of Nonlinear Science* 28.7 (2018), p. 071102 (cit. on p. 11).

[16] Steven H Strogatz and Renato E Mirollo. «Stability of incoherence in a population of coupled oscillators». In: *Journal of Statistical Physics* 63.3-4 (1991), pp. 613–635 (cit. on p. 12).

[17] John David Crawford. «Amplitude expansions for instabilities in populations of globally-coupled oscillators». In: *Journal of statistical physics* 74.5-6 (1994), pp. 1047–1084 (cit. on p. 13).

[18] Helge Dietert. «Stability and bifurcation for the Kuramoto model». In: *Journal de Mathématiques Pures et Appliquées* 105.4 (2016), pp. 451–489 (cit. on p. 13).

[19] Y. Kuramoto. *Chemical Oscillations, Waves, and Turbulence*. Dover Books on Chemistry Series. Dover Publications, 2003. ISBN: 9780486428819. URL: http://books.google.de/books?id=4ADt7smO5Q8C (cit. on p. 13).

[20] Edward Ott and Thomas M Antonsen. «Low dimensional behavior of large systems of globally coupled oscillators». In: *Chaos: An Interdisciplinary Journal of Nonlinear Science* 18.3 (2008), p. 037113 (cit. on p. 14).

[21] Shinya Watanabe and Steven H Strogatz. «Integrability of a globally coupled oscillator array». In: *Physical review letters* 70.16 (1993), p. 2391 (cit. on p. 15).

[22] Yuri L Maistrenko, Oleksandr V Popovych, and Peter A Tass. «Chaotic attractor in the Kuramoto model». In: *International Journal of Bifurcation and Chaos* 15.11 (2005), pp. 3457–3466 (cit. on p. 16).

[23] Yu Maistrenko, Oleksandr Popovych, Oleksandr Burylko, and Peter A Tass. «Mechanism of desynchronization in the finite-dimensional Kuramoto model». In: *Physical review letters* 93.8 (2004), p. 084102 (cit. on p. 16).

[24] Ali Jadbabaie, Nader Motee, and Mauricio Barahona. «On the stability of the Kuramoto model of coupled nonlinear oscillators». In: *Proceedings of the 2004 American Control Conference*. Vol. 5. IEEE. 2004, pp. 4296–4301 (cit. on p. 16).

[25] Bertrand Ottino-Löffler and Steven H Strogatz. «Kuramoto model with uniformly spaced frequencies: Finite-N asymptotics of the locking threshold». In: *Physical Review E* 93.6 (2016), p. 062220 (cit. on p. 16).